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An Extremal Problem of Graphs

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ABSTRACT
We consider the family of graphs with a fixed number of vertices and edges. Among all these graphs, we determine the minimum of the sum of powers α of the vertex degrees, for all $\frac{1}{2}<\alpha<1$, and characterize the optimal graphs. This solves a problem proposed by Dan Ismailescu and Dan Stefanica.

KEYWORDS
minimizer graph; threshold graph; degree sequences

1 Introduction

In this paper, we study an extremal problem in graph theory: among all simple graph on n vertices and e edges, find the ones where the sum of powers α of the vertex degrees is minimum, for all 1/2<α<1. We call these graphs minimizer graphs.

This problem was first considered by Linial and Rozenman[1-2]. In [1], they have showed that, among all simple graph on n vertices and e edges, for the special case when $e=C_k^2$, the one where the sum of the square roots of the vertex degrees is minimum, is a unique graph consisting of a complete graph on k vertices and $n-k$ isolated vertices.

Dan Ismailescu and Dan Stefanica [3] proved that among all simple graph on n vertices and e edges, $e\leqslant C_k^2$, the graph of minimizing $\sum_{i=1}^{n} d_i^{\alpha}$, for all $\alpha\leqslant\frac{1}{2}$, is the unique (up to an isomorphism) graph. They also conjectured that among all graphs with edges, the graph where the sum of powers of the vertex degrees is minimum, for all 1/2<α<1, is unique (up to an isomorphism). In this paper, we prove this conjecture.

Throughout the paper, K_p denotes the complete graph with p vertices, S_p is the edgeless graph with p vertices, and $K_{1,e}$ will indicate the star with e edges.

Let $G(n,e)$ be the family of graphs with n vertices and e edges. We say that $G^*\in G(n,e)$ is a minimizer graph if

$$f(G^*)=\min_{G\in G(n,e)} f(G)$$

where $f(G)=\sum_{i=1}^{n} d_i^{\alpha}$, $\frac{1}{2}<\alpha<1$, with $(d_1,\ldots,d_n)$ the degree sequence of G. We note that this definition does not imply the uniqueness of the minimizer graph.

As proved implicitly in [4], there is a strong connection between minimizer graphs and threshold graphs. A graph G is called a threshold graph if for every three distinct vertices V_i, V_j, V_k of G satisfy the following condition, if $d_i\leqslant d_j$ and V_iV_k is an edge, then V_jV_k is an edge.

2 Some lemmas

Firstly, we need a lemma on an analytic function.

Lemma 2.1 Let $f(x)$ be a decreasing function on $[0,+\infty]$, $f'(x)>0$, for any $0\leqslant a<x\leqslant y<b$, if $\frac{x-a}{b-y}\geqslant 1$, then $f(a)+f(b)\geqslant f(x)+f(y)$.

Proof : Since $f'(x)>0$ on $[0,+\infty]$, $f(x)$ is convex on $[0,+\infty]$, so for any $0\leqslant x_1<x<x_2$,

we have

$$\frac{f(x)-f(x_1)}{x-x_1}<\frac{f(x_2)-f(x_1)}{x_2-x_1}<\frac{f(x_2)-f(x)}{x_2-x}$$

Thus if $0\leqslant a<x<y<b$, then $\frac{f(x)-f(a)}{x-a}<\frac{f(y)-f(x)}{y-x}$, $\frac{f(y)-f(x)}{y-x}<\frac{f(b)-f(y)}{b-y}$, so for any $0\leqslant a<x\leqslant y<b$,

$$\frac{f(x)-f(a)}{x-a}<\frac{f(b)-f(y)}{b-y},$$

i.e.,

$$f(x)-f(a)<\frac{x-a}{b-y}\left(f(b)-f(y)\right)$$

note that $f(b)-f(y)<0$ and $\frac{x-a}{b-y}\geqslant 1$, we get $\frac{x-a}{b-y}\left(f(b)-f(y)\right)\leqslant f(b)-f(y)$,

therefore, $f(x)-f(a)\leqslant f(b)-f(y)$, that is $f(x)+f(y)\leqslant f(a)+f(b)$.

In the following discussions, we denote $g(x)=(x+1)^{\alpha}-x^{\alpha}$, $\frac{1}{2}<\alpha<1$, then $g(0)=1$, $g'(x)=\alpha\left((x+1)^{\alpha-1}-x^{\alpha-1}\right)<0(x>0)$, therefore, $g(x)$ is a decreasing function. Also $g''(x)=\alpha(\alpha-1)\left((x+1)^{\alpha-2}-x^{\alpha-2}\right)>0$ $x\geqslant 0$, from lemma 2.1, for any $0\leqslant a<x\leqslant y<b$, if $\frac{x-a}{b-y}\geqslant 1$, then $g(x)+g(y)\leqslant g(a)+g(b)$.

Now, we show that every minimizer graph must be a threshold graph.

Lemma 2.2 Every minimizer graph is a threshold graph.

Proof : Let G^* be a minimizer graph in $G(n,e)$, and assume that there exist three distinct vertices V_i, V_j, V_k of G^* such that $d_i\leqslant d_j$, V_iV_k is an edge and $V_jV_k\notin E(G^*)$. By replacing V_iV_k with V_jV_k we obtain a new graph $G'\in G(n,e)$. We now prove that $f(G^*)-f(G')>0$, which contradicts the fact that G^* was a minimizer graph.

All the vertices in G^* and G' have the same degrees, except V_i for and V_j. The degrees of V_i and V_j in G', respectively, are $d_i'=d_i-1$ and $d_j'=d_j+1$. Therefore,

$$\begin{aligned}f(G^*)-f(G')&=d_i^{\alpha}+d_j^{\alpha}-(d_i')^{\alpha}-(d_j')^{\alpha}\\&=d_i^{\alpha}-(d_i-1)^{\alpha}+d_j^{\alpha}-(d_j+1)^{\alpha}\\&=g(d_i-1)-g(d_j)>0(d_i\leqslant d_j)。\end{aligned}$$

so V_jV_k is an edge of the graph G^*, and G^* is a threshold graph.

In this paper, we only need the following result, which is a direct consequence of Lemma 2.2.

Corllary 2.3 Assume that a minimizer graph $G\in G(n,e)$ does not have iso- lated vertices, and let V_1 be a vertex of minimal degree of G. Then all the neighbors of V_1 have full degree.

In the first case, we show that any minimizer graph has at least one isolated vertex.

Lemma 2.4 Let $G^*\in G(n,e)$ be a minimizer graph, and assume that $e\leqslant C_{n-1}^2$, then G^* has at least one isolated vertex.

Proof : Once again, we prove the result by contradiction.

Case 1. $C_{n-2}^2<e\leqslant C_{n-1}^2$.

Assume that G^* does not have any isolated vertex, and let V_1 be a vertex of G^* of minimal degree $d_1=k>0$, then V_1 has exactly k neighbors, denoted by $V_{n-k+1}, V_{n-k+2}, \cdots; V_n$. From Corollary 2.3, it follows that all these vertices are of full degree, that is, $d_{n-k+1}=d_{n-k+2}=\cdots=d_n=n-1$. Let $V_2,\cdots;V_{n-k}$ be the other vertices of G^*, of degrees $k\leqslant d_2\leqslant d_3\leqslant\cdots\leqslant d_{n-k}\leqslant n-2$, so $d_1\leqslant d_2\leqslant\cdots\leqslant d_{n-k}\leqslant d_{n-k+1}=\cdots=d_n$.

We construct a graph G' by deleting the k edges from V_1 and replacing them by k edges which were chosen arbitrarily in the subgraph spanned by $V_2,\cdots;V_{n-k}$. We note that this is possible, since the total number of edges is equal to C_{n-1}^2. We discuss the following cases according as the additions of these k edges.

(I) $d_2=e-C_{n-2}^2$

Obviously $n\geqslant 5$, $d_2\geqslant k$, so $d_{n-k+1}=\cdots=d_n=n-1$ in G^*. When $d_2=k$, we may assume that there are t vertices of degree $n-3$ in G^*, so we can add these k edges to the vertices $V_3,\cdots;V_{n-d_2-t}$, then $d_i'=n-3, i=3,\cdots;n-d_2-t$.so

$$\begin{aligned}&f(G^*)-f(G')\\&=[k^{\alpha}+k(n-1)^{\alpha}+d_3^{\alpha}+\cdots+d_{n-k-t}^{\alpha}]-[0^{\alpha}+k(n-2)^{\alpha}+(n-k-t-2)(n-3)^{\alpha}]\\&=(k^{\alpha}-0^{\alpha})+k[(n-1)^{\alpha}-(n-2)^{\alpha}]-[(n-k-t-2)(n-3)^{\alpha}-(d_3^{\alpha}+\cdots+d_{n-k-t}^{\alpha})]\end{aligned}$$

Since $k^{\alpha}-0^{\alpha}=[k^{\alpha}-(k-1)^{\alpha}]+[(k-1)^{\alpha}-(k-2)^{\alpha}]+\cdots+[1^{\alpha}-0^{\alpha}]$. so,

$$f(G^*)-f(G')=g(k-1)+g(k-2)+\cdots+g(0)+kg(n-2)-\sum_{i=3}^{n-k-t}\left[g(n-4)+\cdots+g(d_i)\right]>0$$

When $d_2>k$, we have V_i which is adjacent to the vertex V_2, and $d_i<n-2$, we add some edges to V_2 until $d_i'=n-2$; if V_i is not adjacent to V_2, and $d_i<n-3$, also add edges to V_2 until $d_i'=n-3$, $i=3,\cdots;n-k$. Let $d_3\leqslant\cdots\leqslant d_t<n-3$, $d_{t+1}=n-3$, t is the number of the vertices of $V_3,\cdots;V_{n-k}$ that are adjacent to the vertex V_2 and degree less than $n-2$.so

$$\begin{aligned}&f(G^*)-f(G')\\&=[k^{\alpha}+k(n-1)^{\alpha}+\sum_{i=3}^{n-k}d_i^{\alpha}]-[0^{\alpha}+k(n-2)^{\alpha}+m_1(n-2)^{\alpha}+m_2(n-3)^{\alpha}]\\&=(k^{\alpha}-0^{\alpha})+k[(n-1)^{\alpha}-(n-2)^{\alpha}]-[m_1(n-2)^{\alpha}+m_2(n-3)^{\alpha}-\sum_{i=3}^{n-k}d_i^{\alpha}]\end{aligned}$$

$$\left(m_1+m_2=n-k-2\right)$$
$$=g(k-1)+g(k-2)+\cdots+g(0)+kg(n-2)-\sum_{i=3}^{t}\left[g(n-4)+\cdots+g(d_i)\right]-t_1 g(n-3)$$
$$>0$$

(Ⅱ) $d_2>e-C_{n-2}^2$

Obviously $d_2\geqslant k$.When $d_2=k$, We delete the $d_2-\left(e-C_{n-2}^2\right)=m$ edges from V_2 and add these $m+k$ edges to the vertices $V_3,\cdots,V_{n-k-t}$, (t is the number of the vertices whose degree are $n-3$ in G^*), such that $d_i{}'=n-3$, $i=3,\cdots,n-k-t$. So

$$f\left(G^*\right)-f\left(G'\right)$$
$$=[k^\alpha+k(n-1)^\alpha+d_3^\alpha+\cdots+d_{n-k-t}^\alpha+d_2^\alpha]-[0^\alpha+m(n-3)^\alpha+(k-m)(n-2)^\alpha+(n-k-t-2)(n-3)^\alpha+\left(d_2-m\right)^\alpha]$$
$$=\left(k^\alpha-0^\alpha\right)+(k-m)\,[(n-1)^\alpha-(n-2)^\alpha]+m[(n-1)^\alpha-(n-3)^\alpha]+[d_2^\alpha-\left(d_2-m\right)^\alpha]-[(n-k-t-2)(n-3)^\alpha-(d_3^\alpha+\cdots+ d_{n-k-t}^\alpha)]$$
$$=g(k-1)+g(k-2)+\cdots+g(0)+kg(n-2)+mg(n-3)+g\left(d_2-1\right)+\cdots+g\left(d_2-m\right)-\sum_{i=3}^{n-k-t}\left[g(n-4)+\cdots+g\left(d_i\right)\right]>0$$

When $d_2>k$, We can delete the $d_2-\left(e-C_{n-2}^2\right)=m$ edges from V_2. For example, by deleting edge V_2V_m and add edge V_mV_j $(m,j=3,\cdots,n-k,\ j\neq m)$ we obtain a new image G'. So

$$f\left(G^*\right)-f\left(G'\right)=d_2^\alpha+d_j^\alpha-\left(d_2-1\right)^\alpha-\left(d_j+1\right)^\alpha$$
$$=[d_2^\alpha-\left(d_2-1\right)^\alpha]-[d_j+1^\alpha-d_j^\alpha]$$
$$=g\left(d_2-1\right)-g\left(d_j\right)>0\left(d_j>d_2-1\right)$$

Repeat this step, and the problem can be transformed into either $d_2>k$ in (I) or the situation $d_2=k$ in (II) .

(III) $d_2<e-C_{n-2}^2$

Let $e-C_{n-2}^2-d_2=\varepsilon$ $(1\leqslant\varepsilon\leqslant k)$, We may add $k-\varepsilon$ edges to the vertices $V_3,\cdots,V_{n-k}$(Firstly, we can add edges to V_2 such that $d_2{}'=e-C_{n-2}^2$).

If the vertex V_i is adjacent to V_2 after adding V_2, $d_i<n-2$, then add edges to V_i until $d_i{}'=n-2$; If V_i is not adjacent to V_2 after adding V_2, $d_i<n-3$, we also add edges to V_i until $d_i{}'=n-3$, $i=3,\cdots,n-k$. Let $d_3\leqslant\cdots\leqslant d_t<n-3$, $d_{t+1}=n-3$, t is the number of the vertices of $V_3,\cdots,V_{n-k}$ that are adjacent to the vertex V_2 and degree less than $n-2$. so

$$f\left(G^*\right)-f\left(G'\right)$$
$$=[k^\alpha+k(n-1)^\alpha+\sum_{i=3}^{n-k}d_i^\alpha+d_2^\alpha]-[0^\alpha+k(n-2)^\alpha+m_1(n-2)^\alpha+m_2(n-3)^\alpha+\left(d_2+\varepsilon\right)^\alpha]=\left(k^\alpha-0^\alpha\right)+k[(n-1)^\alpha-(n-2)^\alpha]-$$
$$[m_1(n-2)^\alpha+m_2(n-3)^\alpha-\sum_{i=3}^{n-k}d_i^\alpha]-[\left(d_2+\varepsilon\right)^\alpha-d_2^\alpha]$$

$(m_1+m_2=n-k-2)$

$$=g(k-1)+g(k-2)+\cdots+g(0)+kg(n-2)-\sum_{i=3}^{t}\left[g(n-4)+\cdots+g\left(d_i\right)\right]-\left(t_1+m\right)g(n-3)-[g\left(d_2+\varepsilon-1\right)+\cdots+g\left(d_2\right)]$$
$$>0$$

Case 2. $e\leqslant C_{n-2}^2$

Assume that $C_{l-2}^2<e\leqslant C_{l-1}^2$, $l<n$ is an integer. G^* does not have any isolated vertex, using the same method as in Case 1, we can get the degree sequence $k=d_1\leqslant d_2\leqslant\cdots\leqslant d_{n-k}\leqslant d_{n-k+1}=\cdots=d_n=n-1$.

We construct a graph G' by deleting edges from $V_1,V_2,\cdots,V_i$ $(i=n-l+1)$ and replacing them by edges which were chosen arbitrarily in the subgraph spanned by the other vertices, so that $f\left(G^*\right)-f\left(G'\right)>0$. the proof is similar to that of Case 1.

Thus when $e\leqslant C_{n-2}^2$, we can get a new graph G'such that $f\left(G^*\right)-f\left(G'\right)>0$, contradicting the fact that G^* was chosen to be a minimizer graphs.

Note that there is a close relationship[5,6,3] between minimizing the sum of powers $\alpha\left(\frac{1}{2}<\alpha<1\right)$ of the vertex degrees and maximizing the sum of powers $m(m\geqslant 2)$ of degrees over $G(n,e)$.

Lemma 2.5 [3] Let $m\geqslant 2$ be a fixed integer, and let $G(n,e)$ be the family of graphs with n vertices and e edges. If $e\leqslant n-2$, then $G^*=K_{1,e}\cup S_{n-e-1}$, the star with e edges plus $n-e-1$ isolated vertices, is a maximizer graph for F_m over $G(n,e)$.

Moreover, it is unique with this property, except the case when $m=2$ and $e=3$ (when both $K_{1,3}\cup S_{n-4}$ and $K_3\cup S_{n-3}$ are maximizer graphs).

3 The main result

We are now in a position to prove the uniqueness of the minimizer graph for the sum of powers $\alpha(\frac{1}{2}<\alpha<1)$ of the vertex degrees.

Theorem 3.1Let $G(n,e)$ be the family of graphs with n vertices and e edges, $\frac{1}{2}<\alpha<1$. If $e=C_k^2$, then there is a unique minimizer graph $G_\alpha\in G(n,e)$ corresponding to $f(G)=\sum_{i=1}^{n}d_i^{\ \alpha}$, $\frac{1}{2}<\alpha<1$, and it is isomorphic to the graph with a complete subgraph on k vertices and $n-k$ isolated vertices, i.e., $G_\alpha=K_k\cup S_{n-k}$. Otherwise, let k be the unique positive integer such that $C_{k-1}^2<e<C_k^2$. Once again, there is a unique minimizer graph $G_\alpha\in G(n,e)$ which is isomorphic to the graph with $n-k$ isolated vertices, a complete subgraph K_{k-1}, and one vertex of degree $e-C_{k-1}^2$ connected to vertices of the complete subgraph.

Proof : Let $G\in G(n,e)$ be a minimizer graph. If $e=C_k^2$, we can use Lemma 2.4 to show that all the edges of G belong to a subgraph of G with k vertices. Sinc $e=C_k^2$, that subgraph is complete, and therefore there is exactly one minimizer graph, $G=K_k\cup S_{n-k}$.

Otherwise, let $k\leqslant n$ be the unique positive integer such that

$$C_{k-1}^2<e<C_k^2 \quad ①$$

Using Lemma 2.4 repeatedly, it follows that all the edges of G belong to a subgraph H with k vertices; the other $n-k$ vertices are isolated, i.e., $G=H\cup S_{n-k}$. Since G is a minimizer graph over $G(n,e)$, it follows that $H\in G(k,e)$ must be a minimizer graph over $G(k,e)$.

Let $(d_1,\cdots,d_k)$ be the degree sequence of H, and let $\bar{H}$ be the complement graph of H. Denote by $(\bar{d}_1,\cdots,\bar{d}_k)$, the degree sequence of $\bar{H}$, and by $\bar{e}$, the number of edges of $\bar{H}$. Since $\bar{d}_i=k-1-d_i$, it is easy to see, using ①, that

$$1\leqslant\bar{e}\leqslant k-2 \quad ②$$

We also note that H cannot have isolated vertices. Otherwise, the number of edges e, would be at most C_{k-1}^2, which contradicts ①. Therefore, $d_i\geqslant 1$, which implies that

$$\bar{d}_i=k-1-d_i\leqslant k-2 \quad ③$$

$$f(H)=\sum_{i-1}^{k}d_i^\alpha=\sum_{i=1}^{k}\left(k-1-\bar{d}_i\right)^\alpha=(k-1)^\alpha\sum_{i=1}^{k}\left(1-\frac{\bar{d}_i}{k-1}\right)^\alpha \quad ④$$

$$(1-x)^\alpha=1-\alpha x+\sum_{m=2}^{\infty}\frac{(-1)^m\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!}x^m$$

$$=1-\alpha x-\sum_{m=2}^{\infty}\frac{\alpha(1-\alpha)\cdots(m-\alpha-1)}{m!}x^m \quad ⑤$$

the Taylor expansion is absolutely convergent for every $0<x<1$, and is reduced to a trivial identity for $x=0$. From ③, it follows that $0\leqslant\frac{\bar{d}_i}{k-1}\leqslant\frac{k-2}{k-1}<1$, and therefore we may use the expansion ⑤ for $\left(1-\frac{\bar{d}_i}{k-1}\right)^\alpha(\frac{1}{2}<\alpha<1)$. To simplify notations, let

$c_m=\frac{\alpha(1-\alpha)\cdots(m-\alpha-1)}{m!}$. Then ④ can be written as

$$f(H)=(k-1)^\alpha\sum_{i=1}^{k}\left(1-\frac{\bar{d}_i}{k-1}\right)^\alpha$$

$$=(k-1)^\alpha\sum_{i=1}^{k}[1-\alpha\frac{\bar{d}_i}{(k-1)}-\sum_{m=2}^{\infty}c_m\left(\frac{\bar{d}_i}{k-1}\right)^m]$$

$$=(k-1)^\alpha[k-\alpha\frac{\bar{e}}{(k-1)}-\sum_{m=2}^{\infty}\frac{c_m}{(k-1)^m}\sum_{i=1}^{k}\bar{d}_i^m]$$

$$=(k-1)^\alpha[k-\alpha\frac{\bar{e}}{(k-1)}-\sum_{m=2}^{\infty}\frac{c_m}{(k-1)^m}F_m(\bar{H})] \quad ⑥$$

We recall that $\bar{H}\in G(k,\bar{e})$ and that $\bar{e}\leqslant k-2$. From Lemma2.5, it follows that the star with $\bar{e}$ edges, $K_{1,\bar{e}}$ is a maximizer of $F_m(H')$ over $H'\in G(k,\bar{e})$, for every $m\geqslant 2$. i.e., the graph with $K_{1,\bar{e}}$ and $k-1-\bar{e}$ isolated vertices is a maximizer graph over $G(k,\bar{e})$ (and is unique up to isomorphism for all $m\geqslant 3$). Since all the constants c_m are positive, it follows from ⑥ that $f(H)$ is minimal if and only if $\bar{H}=K_{1,\bar{e}}\cup S_{k-\bar{e}-1}$.

Then, there is a unique minimizer graph H in $G(k,e)$, and therefore a unique minimizer graph $G\in G(n,e)$, with $G=H\cup S_{n-k}$. Since the complement of H is a star with $\bar{e}=C_k^2-e$ edges, the graph H has a complete subgraph K_{k-1}, and one vertex of degree $e - C_{k-1}^2$ connected to vertices of the complete subgraph. The proof is complete.

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